MATH 312
Lecture 21

THE SPECTRAL THEOREM

<u>Recap</u>: Given A, an $n \times n$ matrix,

- Eigenvalues are solutions of $\det(A - \lambda I) = 0$.
- Eigenvectors are nonzero vectors in $N(A - \lambda I)$,
- $A\vec{v} = \lambda \vec{v}$ for eigenval λ, corresponding eigenvector $\vec{v}$
- A is DIAGONALIZABLE if it has n linearly independent eigenvectors
- In this case, $A = SDS^{-1}$, where

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \text{eigenvalue matrix, diagonal,}$$

$$S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \text{eigenvector matrix, invertible.}$$

- <u>WARNING</u> Eigenvalues might be repeated (in which case there is a possibility that A is <u>NOT</u> diagonalizable, eg $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, or eigenvalues may be COMPLEX eg $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

In lecture 19, we diagonalized

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Evals: $\boxed{\begin{matrix} \lambda_1 = 4 \\ \lambda_2 = -1 \end{matrix}}$

Evecs: $\boxed{\begin{matrix} \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{v_2} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \end{matrix}}$

So, $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$,

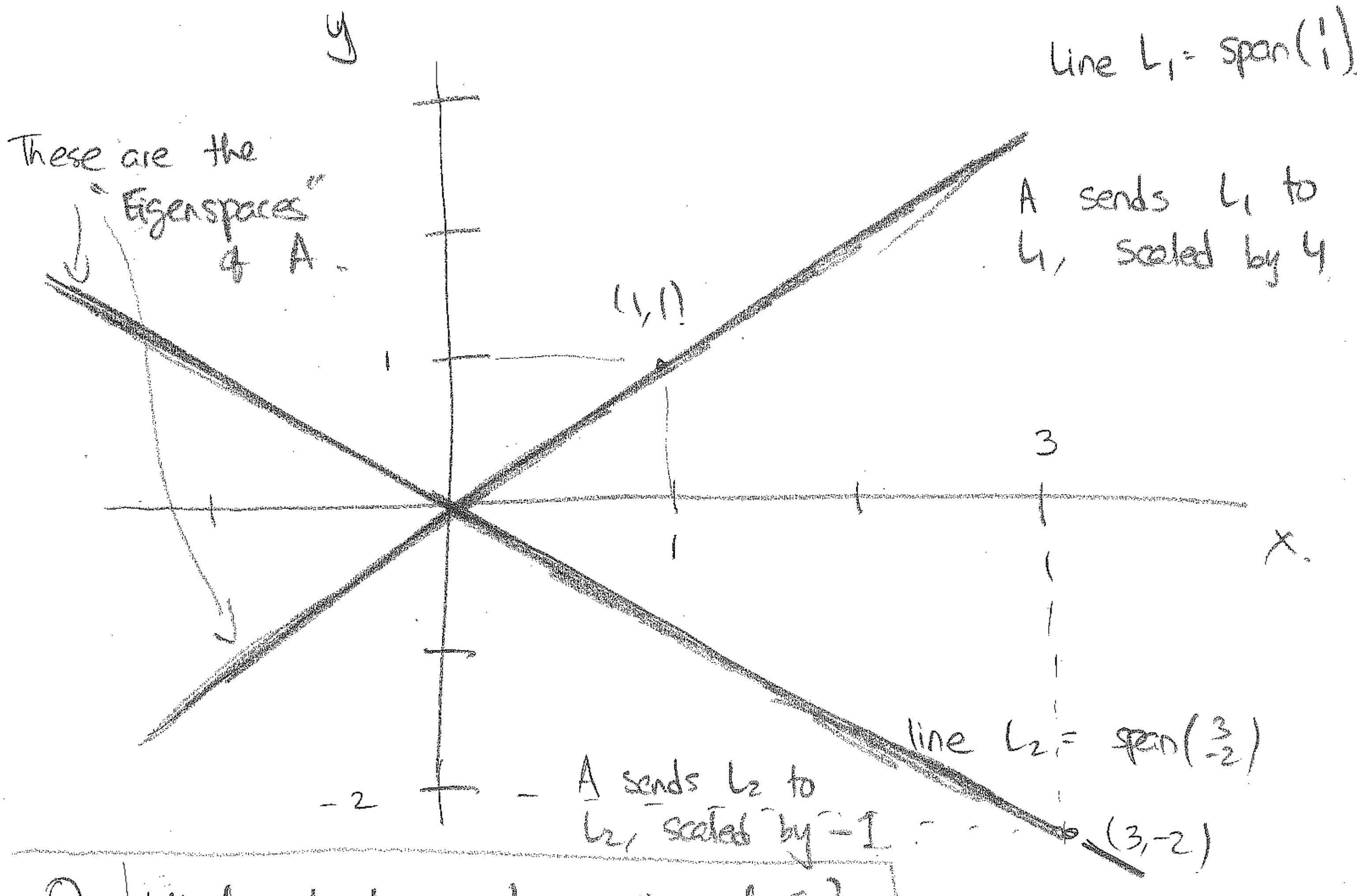
$S = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$

and $S^{-1} = \dfrac{1}{-2-3} \begin{bmatrix} -2 & -3 \\ -1 & 1 \end{bmatrix}$

$= \tfrac{1}{5} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$.

So, $A = \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2/5 & 3/5 \\ 1/5 & -1/5 \end{bmatrix}}_{S^{-1}}$

There's a <u>PICTURE</u> that goes with this:



Q | What about complex eigenvalues?

Complex Numbers : A Crash-Course !

- A complex number z equals $a+bi$, where a and b are real numbers and $i^2 = -1$.
- They can be
  - added: $(a+ib) + (c+id) = (a+c) + (b+d)i$
  - multiplied: $(a+ib)(c+id) = (ac-bd) + (ad+bc)i$
  - <u>conjugated</u>: $\overline{a+ib} = a-ib$.
- If $z = a+ib$, then $z + \bar{z} = 2a$ (purely real)
  and $z - \bar{z} = 2bi$ (" imaginary)

and, most importantly

$$z\bar{z} = (a+ib)(a-ib) = \underline{\underline{a^2+b^2}}$$ real, positive, $\|(a,b)\|^2 \leftarrow$ length-squared.

So $\boxed{\overline{V}^T \cdot V = \|V\| \text{ for complex vectors}}$

Thm If A is $n \times n$ and Real, then its complex eigenvalues, if there are any, must occur in complex conjugate pairs!

Pf if $Ax = \lambda x$, then

$$\overline{Ax} = \overline{\lambda x} \qquad \text{(complex conjugate)}$$

$$\Rightarrow A\bar{x} = \bar{\lambda}\bar{x}.$$

So, the <u>eigenvectors</u> are ALSO conjugate!

The whole picture — eigenvalues, eigenvectors, etc is considerably SIMPLIFIED when the matrix A is SYMMETRIC, i.e, $A = A^T$.

Thm [SPECTRAL THM] If A is a real symmetric $n \times n$ matrix, then it is possible to diagonalize it as

$$A = QDQ^T$$

where: D has only real eigenvalues and Q is orthogonal!

We will spend the rest of lecture proving this theorem. But first, let's test it with a simple example!

Eg

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

$\det(A - \lambda I) = 0$ means

$$(1-\lambda)(5-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 5 - 4 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 1 = 0$$

So,

$$\lambda = \frac{6 \pm \sqrt{36 - 4}}{2}$$

$$= \frac{6 \pm \sqrt{32}}{2}$$

$$= \boxed{3 \pm \sqrt{8}} \leftarrow \text{(Real!)}$$

Eigenvectors: this is painful because of the $\sqrt{8}$ stuff.

$\lambda_1 = 3+\sqrt{8}$

$$A-\lambda_1 I = \begin{bmatrix} -2-\sqrt{8} & 2 \\ 2 & 2-\sqrt{8} \end{bmatrix}$$

An eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ must satisfy

$(-2-\sqrt{8})x + 2y = 0$

$\Rightarrow \cancel{-}(2+\sqrt{8})x = \cancel{-}2y$

So one choice is $\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 2+\sqrt{8} \end{pmatrix} = \vec{v_1}}$

$\lambda_2 = 3-\sqrt{8}$

$$A-\lambda_2 I = \begin{bmatrix} -2+\sqrt{8} & 2 \\ 2 & 2+\sqrt{8} \end{bmatrix}$$

An eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ must satisfy

$(-2+\sqrt{8})x + 2y = 0$

$\Rightarrow \cancel{-}(2-\sqrt{8})x = \cancel{-}2y$

So, one choice is $\boxed{\begin{pmatrix} 2 \\ 2-\sqrt{8} \end{pmatrix} = \vec{v_2}}$

And indeed: $\vec{v_1}$ & $\vec{v_2}$ are orthogonal:

$$v_1^T v_2 = \begin{bmatrix} 2 & 2+\sqrt{8} \end{bmatrix} \begin{bmatrix} 2 \\ 2-\sqrt{8} \end{bmatrix} = 4 + (2+\sqrt{8})(2-\sqrt{8})$$
$$= 4+4-8 = \underline{0}.$$

So, $v_1/\|v_1\|$ and $v_2/\|v_2\|$ would be <u>orthoNORMAL</u> eigenvectors

Let's prove ONE part of the spectral theorem:

<u>Prop</u> If A is real, $n\times n$, symmetric, then its eigenvalues must be Real!

<u>Pf</u> If λ is an eigenvalue, then

$$Av = \lambda v \quad \text{for some } v \neq \vec{0}$$

So, $\overline{Av} = \overline{\lambda v}$ (conjugate both sides)

$\Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$ (A is Real, so $\bar{A} = A$)

(multiply from left by v^T)

So, $v^T A \bar{v} = v^T \bar{\lambda} \bar{v}$

and $v^T A \bar{v} = \bar{\lambda}\, v^T \bar{v}$ (pull out scalar $\bar{\lambda}$)

And so, $\underline{v^T A \bar{v} = \bar{\lambda} \|v\|^2}$ call this $(*)$

Again start with

$$Av = \lambda v$$

So, $v^T A^T = v^T \lambda$ (transpose everything)

which means

$$v^T A = v^T \lambda \quad \text{(A is symmetric, so } A = A^T\text{)}$$

and so, $v^T A \bar{v} = v^T \lambda \bar{v}$ (mult on right by $\bar{v}$)

$\Rightarrow v^T A \bar{v} = \lambda v^T \bar{v}$ (pull out scalar λ)

$\Rightarrow \underline{v^T A \bar{v} = \lambda \|v\|^2}$

Comparing with $(*)$, we get

$$\bar{\lambda} \|v\|^2 = \lambda \|v\|^2 .$$

Since v is an eigenvector, it is NOT zero, so we can cancel $\|v\|^2 \neq 0$ from both sides to get

$$\bar{\lambda} = \lambda .$$

Since λ equals its own conjugate, it must be Real: i.e., $a+ib = a-ib$ means $b=0$!

We've proved half of the Spectral theorem:
Now we want to show that the eigenvectors are orthogonal (so dividing by length makes them orthoNORMAL).

To make life easier, we assume that the eigenvalues are <u>DISTINCT</u>. The Spectral Theorem works even when they are Not, but the proof gets more complicated.

Prop Let A be a Real, Symmetric $n \times n$ matrix with distinct eigenvalues. Then, the corresponding eigenvectors are ORTHOGONAL!

Pf Let λ and K be two eigenvalues with corresponding eigenvectors v and u; we want to show that $v^T u = 0$.

First, $Av = \lambda v$, so

$$v^T A^T = v^T \lambda \quad \text{(transpose both sides!)}$$

which means, $\boxed{v^T A = v^T \lambda}$ (since $A = A^T$)

Second,

$$Au = Ku$$

so $v^T A u = v^T K u$ [multiply both sides by v^T from left]

This implies

$$\Rightarrow v^T \lambda u = v^T K u \quad \text{[pull out scalars]}$$

$$\Rightarrow \lambda v^T u = K v^T u$$

$$\Rightarrow (\lambda - K)\, v^T u = 0.$$

Since $\lambda \neq K$ by the distinct eigenvalues assumption, we must get $\underline{v^T u = 0}$ as desired.